

$$f(n) = \begin{cases} \cot \frac{\pi n}{4} & ; n < 1 \\ \sqrt{n^2 + 1} & ; n > 1 \end{cases}$$

$$f\left(\frac{1}{\sqrt{3}}\right) = \cot \frac{\pi \frac{1}{\sqrt{3}}}{4} = \cot \frac{\pi}{4} = \sqrt{3}$$

$$\Rightarrow f \circ f\left(\frac{1}{\sqrt{3}}\right) = f(\sqrt{3}) = \sqrt{(\sqrt{3})^2 + 1} = \boxed{2}$$

$$\text{الف) } f\left(\frac{1}{n}\right) = \sqrt{\frac{2n-1}{n^2}} \quad g(n) = 2 \cos^2 n$$

$$g\left(\frac{\pi}{4}\right) = 2 \cos^2 \frac{\pi}{4} = \frac{1}{2} \rightarrow f \circ g = f\left(\frac{1}{2}\right) = \sqrt{\frac{2 \cdot \frac{1}{2} - 1}{\left(\frac{1}{2}\right)^2}} = \boxed{\frac{\sqrt{2}}{2}}$$

$$\text{ب) } f(n) = [n] \quad g(n) = \frac{n}{1-n}$$

$$\Rightarrow f \circ g(n) = \left[\frac{n}{1-n} \right] = \left[\frac{n-1+1}{1-n} \right] = \left[\frac{1}{1-n} \right] - 1 \rightarrow f \circ g(\sqrt{2}) = \left[\frac{1}{1-\sqrt{2}} \right] - 1 = [\sqrt{2}-1] - 1 = -2$$

$$\Rightarrow f \circ g(\sqrt{2}) = [\sqrt{2}] - 2 = \boxed{-1}$$

$$f(n) = \sin n$$

$$g(n) = n \sqrt{1-n^2}$$

$$f\left(\frac{\pi}{4}\right) = \sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}$$

$$g \circ f\left(\frac{\pi}{4}\right) = g\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2} \sqrt{1 - \left(\frac{\sqrt{2}}{2}\right)^2} = \frac{\sqrt{2}}{2} \times \frac{1}{\sqrt{2}} \Rightarrow \boxed{f \circ g\left(\frac{\pi}{4}\right) = \frac{1}{2}}$$

$$f(n) = \{(4, \omega), (4, \omega), (1, 12), (10, 13)\}$$

$$g(n) = \{(4, 4), (2, 4), (4, 1), (1, 10)\}$$

$$\text{الف) } f \circ g(n) = \{(4, \omega), (2, \omega), (4, 12), (1, 12)\}$$

$$\text{ب) } g \circ f(n) = \emptyset$$

$$\text{ج) } f \circ f(n) = \emptyset$$

$$\text{د) } g \circ g(n) = \{(4, 1), (2, 4), (4, 10)\}$$

$$f = \{(2, 1), (3, 2), (4, \omega), (1, 1)\}$$

$$g = \{(1, 2), (3, 1), (a, 3), (b, 1)\}$$

$$(4, 2) \in f \circ g \rightarrow f(n) = 2 \rightarrow n = 3 \rightarrow f(g) = 2 \rightarrow \boxed{g = a \in f}$$

$$(4, 1) \in g \circ f \rightarrow f(4) = a \rightarrow f(\omega) = 1 \rightarrow \boxed{\omega = b}$$

$$\Rightarrow \boxed{(a, b) = (f, \omega)}$$

$$\begin{aligned} f(f(n)) &= r_{n+r} & g(r_{n+r}) &= r_{n-r} \\ f(n) &= an+b \rightarrow f(f(n)) &= a^2n+ab+b &= r_{n+r} \rightarrow a^2=r \rightarrow a=\pm r \\ &\rightarrow a=r \rightarrow b=1 \rightarrow f(n) &= r_{n+1} & \Rightarrow f(-1) = -1 \\ &\rightarrow a=-r \rightarrow b=-r \rightarrow f(n) &= -r_{n-r} \\ g \circ f(-1) &= g(-1) \rightarrow -1 &= r_{n+r} \rightarrow n=-r \rightarrow g(-1) &= r_{n(-r)-r} = \boxed{-1} \end{aligned}$$

$$f(n) = \sqrt{n+1} \quad g(n) = \frac{1}{n^2-n}$$

$$D_{g \circ f} = \{n \in D_f : f(n) \in D_g\}$$

$$D_f: n+1 \geq 0 \rightarrow n \in \mathbb{R} \quad \text{für } n \geq -1$$

$$D_g: x \neq 0, \pm 1 \rightarrow \begin{cases} f(n) \neq 0 \rightarrow n \in \mathbb{R}^+ \\ f(n) \neq \pm 1 \rightarrow n \neq 1 \end{cases} \rightarrow \boxed{D_{g \circ f} = \mathbb{R}^+ - \{1\}}$$

$$f(x) = \sqrt{1-x^2} \quad g(x) = \sqrt{x}$$

$$D_f: -1 \leq x \leq 1 \quad D_{f \circ g} = D_f \cap D_g = [0, 1]$$

$$D_g: x \geq 0$$

$$D_{(f \circ g)} \text{ of } : \{x \in D_f, f(x) \in D_{f \circ g}\} \rightarrow \begin{aligned} & -1 \leq x \leq 1 \quad \textcircled{1} \\ & 0 \leq \sqrt{1-x^2} \leq 1 \rightarrow 1-x^2 \leq 1 \rightarrow x^2 \geq 0 \end{aligned} \quad \textcircled{2}$$

$$\therefore D_{(f \circ g)} \text{ of } = [-1, 1]$$

$$\text{ii)} \quad f\left(\frac{r_{m+1}}{n-r}\right) = r_m + \Delta \quad \Rightarrow \quad f(t) = r\left(\frac{r_t+1}{t-r}\right) + \Delta = \frac{r^2 t - 11}{t-r} \quad \Rightarrow \quad \boxed{f(n) = \frac{r^2 n - 11}{n-r}}$$

$$\text{c)} \quad f\left(n + \frac{1}{n}\right) = n^r + \frac{1}{n^r} = \left(n + \frac{1}{n}\right) \left(n^r + \frac{1}{n^r} - 1\right) = \left(n + \frac{1}{n}\right) \left(\left(n + \frac{1}{n}\right)^r - r\right)$$

$$\underline{n + \frac{1}{n} = t}, \quad f(t) = t(t^r - r) \rightarrow f(t) = t^{r+1} - rt \rightarrow \boxed{f(n) = n^{r+1} - rn}$$

$$\begin{aligned} g(n) &= a(n-1)(n-\sqrt{r}) \\ f(n) &= n\sqrt{r} \\ g \circ f(n) &= a(n\sqrt{r}-1)(n\sqrt{r}-\sqrt{r}) \xrightarrow{\text{عوضه و ساده کن}} \begin{aligned} n\sqrt{r}-1 &\rightarrow n+1 \\ n\sqrt{r}-\sqrt{r} &\rightarrow n+1 \end{aligned} \\ \Rightarrow r-1 &= 1 \end{aligned}$$