۱)

$$f(x)=\begin{cases}\cot\dfrac{\pi x}{4} & ; x\le 1\\[2mm] \sqrt{x^2+1} & ; x>1\end{cases}$$

$$f(f(\tfrac{2}{3})) \Rightarrow f(\tfrac{2}{3}) = \cot\frac{2\pi}{12} = \cot\frac{\pi}{6} = \sqrt{3}$$

$$f(\sqrt{3}) = \sqrt{(\sqrt{3})^2+1} = \boxed{2}$$

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۲)

الف) $f(\frac{1}{x}) = \sqrt{\dfrac{2x-1}{x^2}}$, $g(x) = 2\cos^2 x$, $fog(\frac{\pi}{3})$

$$f(\tfrac{1}{x}) = \sqrt{\frac{2x-1}{x^2}} \rightarrow \frac{1}{x} = t \;;\; \frac{2x-1}{x^2} = \frac{2x}{x^2} - \frac{1}{x^2} = \frac{2}{x} - \frac{1}{x^2} = 2t - t^2$$

$$\Rightarrow f(t) = \sqrt{2t - t^2} \Rightarrow f(x) = \sqrt{2x - x^2}$$

$$f(\underbrace{g(\tfrac{\pi}{3})}_{\frac{1}{2}}) \Rightarrow g(x) = 2\cos^2 x \xrightarrow{x=\frac{\pi}{3}} 2\times\frac{1}{4} = \frac{1}{2}$$

$$\Rightarrow f(\tfrac{1}{2}) = \sqrt{2\times\frac{1}{2} - (\tfrac{1}{2})^2} = \sqrt{1-\frac{1}{4}} = \sqrt{\frac{3}{4}} = \boxed{\frac{\sqrt{3}}{2}}$$

ب) $f(x) = [x]$, $g(x) = \dfrac{x}{1-x}$, $fog(\sqrt{2})$

$$f(\underbrace{g(\sqrt{2})}_{-2-\sqrt{2}}) \Rightarrow g(\sqrt{2}) = \frac{\sqrt{2}}{1-\sqrt{2}} \times \frac{1+\sqrt{2}}{1+\sqrt{2}} = \frac{\sqrt{2}+2}{-1} = -2-\sqrt{2}$$

$$f(-2-\sqrt{2}) \Rightarrow [\underbrace{-2-\sqrt{2}}_{-3,4\dots}] = \boxed{-4}$$

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۳)

$f(x) = \sin x$ $\quad g(x) = x\sqrt{1-x^2}$ $\quad gof(\frac{\pi}{4}) = g(f(\frac{\pi}{4})) = ?$

$$f(\tfrac{\pi}{4}) = \sin(\tfrac{\pi}{4}) = \frac{\sqrt{2}}{2}$$

$$g(\tfrac{\sqrt{2}}{2}) = \frac{\sqrt{2}}{2}\sqrt{\underbrace{1-(\tfrac{\sqrt{2}}{2})^2}_{\frac{1}{2}}} = \frac{\sqrt{2}}{2}\times\frac{1}{\sqrt{2}} = \boxed{\frac{1}{2}}$$

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۴)

$f(x) = \{(4,5),(6,5),(8,12),(10,12)\}$ $\quad g(x) = \{(4,6),(2,4),(6,8),(8,10)\}$

الف) $fog(x)$:

$$4 \xrightarrow{g(x)} 6 \xrightarrow{f(x)} 5$$
$$2 \xrightarrow{g(x)} 4 \xrightarrow{f(x)} 5$$
$$6 \xrightarrow{g(x)} 8 \xrightarrow{f(x)} 12$$
$$8 \xrightarrow{g(x)} 10 \xrightarrow{f(x)} 12$$

$$\boxed{fog(x) = \{(4,5),(2,5),(6,12),(8,12)\}}$$

ب) $gof(x)$:

$$4 \xrightarrow{f(x)} 5 \longrightarrow \times$$
$$6 \xrightarrow{f(x)} 5 \longrightarrow \times$$
$$8 \xrightarrow{f(x)} 12 \longrightarrow \times$$
$$10 \xrightarrow{f(x)} 12 \longrightarrow \times$$

$$\boxed{gof(x) = \varnothing}$$

ج) $fof(x)$:

$$4 \xrightarrow{f(x)} 5 \longrightarrow \times$$
$$6 \xrightarrow{f(x)} 5 \longrightarrow \times$$
$$8 \xrightarrow{f(x)} 12 \longrightarrow \times$$
$$10 \xrightarrow{f(x)} 12 \longrightarrow \times$$

$$\boxed{fof(x) = \varnothing}$$

د) $gog(x)$:

$$4 \xrightarrow{g(x)} 6 \xrightarrow{g(x)} 8$$
$$2 \xrightarrow{g(x)} 4 \xrightarrow{g(x)} 6$$
$$6 \xrightarrow{g(x)} 8 \xrightarrow{g(x)} 10$$
$$8 \xrightarrow{g(x)} 10 \xrightarrow{g(x)} \times$$

$$\boxed{gog(x) = \{(4,8),(2,6),(6,10)\}}$$

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۵)

$f = \{(2,1),(3,2),(4,5),(1,7)\}$ $\quad g = \{(1,2),(3,1),(a,3),(b,1)\}$ $\quad (4,2)\in fog$ $\quad (4,1)\in gof$ $\quad (a,b) = ?$

$$f(g(4)) = 2 \Rightarrow \square \xrightarrow{g(x)} 3 \xrightarrow{f(x)} 2 \rightarrow \boxed{\square = a = 4}$$

$$g(f(4)) = 1 \Rightarrow 4 \xrightarrow{f(x)} 5 \xrightarrow{g(x)} 1 \longrightarrow$$

در $g(x)$ ۲ جفت داریم که خروجی شان ۱ است : $(3,1)$ $(b,1)$

اما می خواهیم ورودی ۵ باشد $\Leftarrow$ $\boxed{b=5}$

$$(a,b) \Rightarrow \boxed{(4,5)}$$

۶

$f(f(x))=4x+3 \quad g(2x+3)=3x-2 \quad gof(-1)=?$

$\llcorner f(x)=ax+b \qquad \llcorner g(x)=cx+d$

$f(f(x))=a(ax+b)+b=a^2x+ab+b$

$a^2x+ab+b=4x+3 \to \begin{cases} a^2=4 \to a=\pm 2 \\ ab+b=3 \xrightarrow{a=2} b=1 \ : \ f(x)=2x+1 \\ \quad \xrightarrow{a=-2} b=-3 \ ; \ f(x)=-2x-3 \end{cases}$

$g(2x+3)=3x-2$

$c(2x+3)+d \Rightarrow 2cx+3c+d=3x-2 \quad \begin{cases} 2c=3 \to c=\frac{3}{2} \\ 3c+d=-2 \ ; \ 3(\frac{3}{2})+d=-2 \to d=-\frac{13}{2} \end{cases}$

$gof(-1) \Rightarrow f(x): 2x+1 \to f(-1)=2(-1)+1=-1$

$g(-1)=\frac{3}{2}(-1)-\frac{13}{2}=\frac{-16}{2}=\boxed{-8}$

$gof(-1) \Rightarrow f(x)=-2x-3 \to f(-1)=-2(-1)-3=-1$

$g(-1)=\frac{3}{2}(-1)-\frac{13}{2}=\boxed{-8}$

$\boxed{gof(-1)=-8}$

۷

$f(x)=\sqrt{x+|x|} \quad g(x)=\frac{1}{x^2-4x} \qquad D_{gof}:\{x\in D_f \ ; \ f(x)\in D_g\}$

$\sqrt{x+|x|} \to x+|x|\geq 0 \Rightarrow \begin{cases} 0 \ ; \ x<0 \\ \sqrt{2x} \ ; \ x\geq 0 \end{cases} \Rightarrow D_f=\mathbb{R}$

$g(x)=\frac{1}{x^2-4x} \Rightarrow x^2-4x\neq 0 \to x(x-4)\neq 0 \quad x\neq 0,4$

$\{x\in\mathbb{R} \cap \underbrace{\sqrt{x+|x|}\neq\{0,4\}}\}$

$\llcorner \sqrt{x+|x|}=0 \to x\leq 0$ غیرقابل قبول

$\llcorner \sqrt{x+|x|}=4 \to \sqrt{2x}=4 \to 2x=16 \to x\neq 8$

$\Rightarrow D_{gof}=\boxed{(0,+\infty)-\{8\}}$ یا $\boxed{(0,8)\cup(8,+\infty)}$

۸

$f(x)=\sqrt{1-x^2} \ , \ g(x)=\sqrt{x} \quad D(f+g)of$

$D_f \Rightarrow 1-x^2\geq 0 \to x^2\leq 1 \to x\in[-1,1]$

$D_g \Rightarrow \sqrt{x} \to x\geq 0 \to x\in[0,+\infty)$

$D(f+g)=D_f\cap D_g \to [-1,1]\cap[0,+\infty)=[0,1]$

$D(f+g)of=\{x\in D_f : f(x)\in D(f+g)\}=\{-1\leq x\leq 1 : 0\leq\sqrt{1-x^2}\leq 1\} \Rightarrow \boxed{D(f+g)of=[-1,1]}$

اگر دقت کنیم می بینیم به ازای $[-1,0]$ برای $\sqrt{1-x^2}$ هم جواب می دهد

۹

الف) $f(\frac{3x+1}{x-2})=4x+5$

$\frac{3x+1}{x-2}=t \to 3x+1=tx-2t \to 2t+1=tx-3x \Rightarrow 2t+1=x(t-3) \Rightarrow x=\frac{2t+1}{t-3}$

$\to 4(\frac{2t+1}{t-3})+5=\frac{8t+4}{t-3}+5=\frac{8t+4+5t-15}{t-3}=\frac{13t-11}{t-3} \Rightarrow \boxed{f(x)=\frac{13x-11}{x-3}}$

ب) $f(x+\frac{1}{x})=x^3+\frac{1}{x^3}$

$x+\frac{1}{x}=t \ ; \ x^3+\frac{1}{x^3}=(x+\frac{1}{x})^3-3(x+\frac{1}{x})=t^3-3t$

$f(t)=t^3-3t \to \boxed{f(x)=x^3-3x}$

15

$g(1)=0$ و $g(2\sqrt{2})=0$

$g(f(x))=0 \longrightarrow f(x)=1$ یا $f(x)=2\sqrt{2}$

$f(x)=x\sqrt{x}=1 \rightarrow x=1$

$f(x)=x\sqrt{x}=2\sqrt{2} \rightarrow x=2$

بنابراین $g\circ f(x)$ در $x=1$ و 2 محور x ها را قطع می کند

$\boxed{2-1=1}$