(۱)

$\log_n^m = a$

$\log_{mn}^{m^2 n} = b \Rightarrow \dfrac{\log_n^{m^2 n}}{\log_n^{mn}} = \dfrac{\cancel{\log_n^n}^{1} + 2\log_n^m}{\log_n^m + \cancel{\log_n^n}^{1}} = \dfrac{2a+1}{a+1} \xrightarrow{a>0} \dfrac{a+1}{a+1} < \dfrac{2a+1}{a+1} < \dfrac{2a+2}{a+1} \Rightarrow 1 < \dfrac{2a+1}{a+1} < 2 \Rightarrow [b]=1$

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(۲)

الف) $y = \sqrt{\dfrac{x}{\log_{\frac{1}{2}}^x}} \Rightarrow \begin{cases} x>0 \\ \log_{\frac{1}{2}}^x \neq 0 \Rightarrow x \neq 1 \\ \dfrac{x}{\log_{\frac{1}{2}}^x} \geq 0 \quad (x \to x=0,\ \log_{\frac{1}{2}}^x \to x=1) \Rightarrow (0,1) \end{cases} \xrightarrow{\cap} D_f = (0,1)$

ب) $y = \dfrac{\log_4^{(x^2-x-2)}}{\sqrt{x^2-1}+1} \Rightarrow \begin{cases} x^2-x-2>0 \Rightarrow (-\infty,-1) \cup (2,+\infty) \quad (x=-1,\ x=2) \\ x^2-1 \geq 0 \Rightarrow x^2 \geq 1 \Rightarrow x \geq 1 \text{ یا } x \leq -1 \\ \sqrt{x^2-1}+1 \neq 0 \Rightarrow \sqrt{x^2-1} \neq -1 \Rightarrow \mathbb{R} \quad \text{همواره برقرار است} \end{cases} \xrightarrow{\cap} (-\infty,-1) \cup (2,+\infty)$

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(۳)

$2\log_x^9 + \log_a^{\sqrt{x}} = 2 \xrightarrow{x=9} 2\log_{3^2}^a + \log_a^3 = 2$

$\cancel{2}\,\tfrac{1}{\cancel{2}}\log_3^a + \log_3^a = 2 \xrightarrow{\log_3^a = t} \dfrac{1}{t} + t = 2 \xrightarrow{t \neq 0} 1 + t^2 = 2t$

$t^2 - 2t + 1 = 0$

$t=1 \Rightarrow \log_3^a = 1 \Rightarrow a=3$

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(۴)

$(\log^{\frac{5}{3}})x^2 + (\log^9)x - \log^{15} = 0$

$(\underbrace{\log^5}_{0/7} - \underbrace{\log^3}_{0/4})x^2 + (2\underbrace{\log^3}_{0/4})x - (\underbrace{\underbrace{\log^3}_{0/4} + \underbrace{\log^5}_{0/7}}_{1/1}) \Rightarrow \log^5 = \log^{\frac{10}{2}} = \log^{10} - \log^2 \Rightarrow \log^5 = 1 - \underbrace{\log^2}_{0/3} = 0/7$

$0/3x^2 + 0/8x - 1/1 = 0 \Rightarrow 3x^2 + 8x - 11 = 0 \quad \begin{cases} x=1 \\ x = -\frac{11}{3} \end{cases} \Rightarrow |x_1 - x_2| = |1 + \tfrac{11}{3}| = \tfrac{14}{3}$

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(۵)

$\log_2^7 = 2/8 \quad , \quad \log_5^3 = 0/5$

$\log_{21}^{10} = ? \Rightarrow \log_{21}^{10} = \dfrac{1}{\log_{10}^{21}}$

$\log_{10}^{21} = \underbrace{\log_{10}^3}_{\frac{1}{3}} + \underbrace{\log_{10}^7}_{\frac{2/8}{3}} \Longrightarrow \dfrac{1}{3} + \dfrac{2/8}{3} = \dfrac{3/8}{3} = \dfrac{38}{30}$

$\log_{10}^3 = \dfrac{\log_5^3}{\log_5^{10}} = \dfrac{0/5}{\underbrace{\log_5^5}_{1} + \underbrace{\log_5^2}_{0/5}} = \dfrac{1}{3}$

$\log_{10}^7 = \dfrac{\log_2^7}{\log_2^{10}} = \dfrac{2/8}{\underbrace{\log_2^2}_{1} + \underbrace{\log_2^5}_{2}} = \dfrac{2/8}{3}$

$\log_2^5 = \dfrac{1}{\log_5^2} = \dfrac{1}{\frac{1}{2}} = 2$

$\log_{21}^{10} = \dfrac{1}{\frac{38}{30}} = \dfrac{30}{38} = \dfrac{15}{19}$

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(۶)

$\log_{15}^6 = \dfrac{1}{\log_6^{15}}$

$\log_6^{15} = \log_6^3 + \log_6^5 = \dfrac{8}{13} + \dfrac{12}{13} = \dfrac{20}{13}$

$\log_6^3 = \dfrac{\log_2^3}{\log_2^6} = \dfrac{1/6}{\underbrace{\log_2^2}_{1} + \underbrace{\log_2^3}_{1/6}} = \dfrac{1/6}{2/6} = \dfrac{16}{26} = \dfrac{8}{13}$

$\log_6^5 = \dfrac{\log_3^5}{\log_3^6} = \dfrac{1/5}{\log_3^2 + \underbrace{\log_3^3}_{1}} = \dfrac{\frac{3}{2}}{\frac{13}{8}} = \dfrac{12}{13}$

$\log_3^2 = \dfrac{1}{\log_2^3} = \dfrac{1}{1/6} = \dfrac{10}{16} = \dfrac{5}{8}$

$\log_{15}^6 = \dfrac{1}{\frac{20}{13}} = \dfrac{13}{20}$

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(۷)

$\log_4^{12} = \log_4^3 + \underbrace{\log_4^4}_{1}$

$\log_8^{18} = \dfrac{\log_4^{18}}{\log_4^8} = \dfrac{\log_4^2 + \log_4^9}{\frac{3}{2}\underbrace{\log_4^4}_{1}} = \dfrac{\frac{1}{2}\log_2^2 + 2\log_4^3}{\frac{3}{2}} = \dfrac{\frac{1}{2} + 2\log_4^3}{\frac{3}{2}} = m$

$\frac{3}{2}m = \frac{1}{2} + 2\log_4^3 \Rightarrow \log_4^3 = \dfrac{\frac{3m-1}{2}}{2}$

$\log_4^3 = \dfrac{3m-1}{4}$

$\log_4^{12} = \dfrac{3m-1}{4} + 1 = \dfrac{3m+3}{4}$

(۸)

$$\left(\frac{۲۵}{۸}\right)^{x^۲} = (۰/۴)^{۲x-۱} \Rightarrow \left(\frac{۲}{۵}\right)^{-۳x^۲} = \left(\frac{۲}{۵}\right)^{۲x-۱} \Rightarrow -۳x^۲ = ۲x-۱ \Rightarrow ۳x^۲+۲x-۱=۰ < \begin{array}{l} x_۱=-۱ \Rightarrow ۹x+۱=-۸ \text{ غ ق ق} \\ x_۲=\frac{۱}{۳} \Rightarrow ۹x+۱=۴ \text{ ق ق} \end{array}$$

$$\Downarrow \quad \left(\frac{۵^۲}{۲^۳}\right)^{x^۲} = \left(\frac{۲}{۵}\right)^{-۳x^۲}$$

$$\log_{۸}^{(۹x+۱)} \Rightarrow ۹x+۱>۰ \Rightarrow x=\frac{۱}{۳} \text{ ق ق},\ x=-۱ \text{ غ ق ق} \Rightarrow \log_{۸}^{۴} = \frac{۲}{۳}$$

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(۹)

$$\log_{۸}^{b} = \frac{۲}{۳}(۱+a) \Rightarrow \log_{۸}^{b} = \frac{۲}{۳}(\underbrace{\log_{۲}^{۲} + \log_{۲}^{۳}}_{\log_{۲}^{۶}}) \Rightarrow \log_{۸}^{b} = \log_{۸}^{۳۶} \Rightarrow b=۳۶$$

$$\log^{(۳b-۸)} = \log_{۱۰}^{۱۰۰} = ۲$$

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(۱۰)

$$\alpha+\beta = S = \frac{-b}{۴a} = \frac{b}{۴a} \xRightarrow{\text{فرض}} \frac{۴a}{b} = \log^{۴} \Rightarrow \frac{a}{b} = \frac{۱}{۴}\log^{۴} \Rightarrow \frac{a}{b} = \frac{۱}{۲}\log^{۲} \Rightarrow a = \frac{b}{۲}\log^{۲}$$

$$a+c=۲a \Rightarrow c=۲a-b \Rightarrow c = b\log^{۲} - b,$$

$$c = b(\log^{۲}-۱) \Rightarrow c = -b(\underset{\log^{۱۰}}{\cancel{۱}} - \log^{۲}) \Rightarrow c = -b(\log^{۵})$$

$$\frac{c}{a} = \frac{-\cancel{b}\log^{۵}}{\frac{\cancel{b}}{۲}\log^{۲}} = -۲\log_{۲}^{۵} \Rightarrow \left(\frac{۱}{\sqrt[۸]{۲}}\right)^{\frac{c}{a}} = (۲^{-\frac{۱}{۸}})^{-۲\log_{۲}^{۵}} = (۲^{\frac{۱}{۴}})^{\log_{۲}^{۵}} \Rightarrow ۵^{\log_{۲}^{۲^{\frac{۱}{۴}}}} = ۵^{\frac{۱}{۴}} = \sqrt[۴]{۵}$$