به نام خدا

کیوان مکارم منش — تکلیف شماره ۱۷

۱۸٫۵

۱) می‌دانیم $\tan^2 x + 1 = \dfrac{1}{\cos^2 x} \rightarrow 8\cos x - \dfrac{1}{\cos^2 x} + \cancel{1} = \cancel{1}$

$\rightarrow 8\cos x = \dfrac{1}{\cos^2 x} \xrightarrow[\times \cos^2 x]{\cos x \neq 0} 8\cos^3 x = 1 \rightarrow \cos^3 x = \dfrac{1}{8} \xrightarrow{\sqrt[3]{}} \cos x = \dfrac{1}{2}$

$\rightarrow \cos x = \dfrac{1}{2} = \cos\left(\dfrac{\pi}{3}\right) \rightarrow x = 2k\pi + \dfrac{\pi}{3} \rightarrow k=0 \Rightarrow x = \left\{\dfrac{\pi}{3}\right\}$ ✓

$x = 2k\pi - \dfrac{\pi}{3} \xrightarrow{x \in [0, 2\pi]} k=1 \Rightarrow x = \left\{2\pi - \dfrac{\pi}{3}\right\}$ ✓

درنتیجه ۲ جواب $\left\{\dfrac{\pi}{3}, \dfrac{5\pi}{3}\right\}$ را در این بازه داریم.

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۲) $5(1-\cos^2 x) + 2(4\cos^3 x - 3\cos x) = -2 \rightarrow 5 - 5\cos^2 x + 8\cos^3 x - 6\cos x = -2$

$\xrightarrow{\cos x = t} 8t^3 - 5t^2 - 6t + 7 = 0 \xrightarrow{\text{تجزیه}} (t+1)(\underbrace{8t^2 - 13t + 7}_{\Delta < 0}) = 0 \rightarrow t = -1$

$\cos x = -1 \rightarrow x = 2k\pi + \pi$

$\xrightarrow{x \in [-\pi, \pi]} x = \{\pi, -\pi\} \rightarrow$ ۲ جواب دارد ✓

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۳) $2\sin x(\cos 2x) + \sin x = 1 \xrightarrow{\cos 2x = 1 - 2\sin^2 x} 2\sin x(1 - 2\sin^2 x) + \sin x = 1$

$\xrightarrow{\sin x = t} 2t - 4t^3 + t = 1 \rightarrow 4t^3 - 3t + 1 = 0 \xrightarrow[\text{بخش‌پذیر است}]{t+1}$

$\rightarrow (t+1)(4t^2 - 4t + 1) = 0 \xrightarrow{t+1=0}$ ① $\sin x = -1 \rightarrow x = 2k\pi + \dfrac{3\pi}{2}$

② $4t^2 - 4t + 1 = 0 \rightarrow t = \dfrac{1}{2} \rightarrow$ ② $\sin x = \dfrac{1}{2} \rightarrow x = 2k\pi + \dfrac{\pi}{6}$

$x = 2k\pi + \dfrac{5\pi}{6}$

$\xrightarrow{x \in [0, 2\pi]} x = \left\{\dfrac{3\pi}{2}, \dfrac{\pi}{6}, \dfrac{5\pi}{6}\right\} \xrightarrow{\text{مجموع}} \dfrac{\pi}{6} + \dfrac{5\pi}{6} + \dfrac{3\pi}{2} = \pi + \dfrac{3\pi}{2} = \dfrac{5\pi}{2}$ ✓

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۴) $\dfrac{1}{\sin\left(\frac{\pi}{2} + 2x\right)} + \dfrac{1}{\cos\left(\frac{\pi}{2} + 4x\right)} = 0 \rightarrow \dfrac{1}{\cos 2x} + \dfrac{-1}{\underset{2\sin 2x \cos 2x}{\sin 4x}} = 0$

$= \dfrac{2\sin 2x - 1}{2\sin 2x \cos 2x} = 0 \Rightarrow \dfrac{2\sin 2x - 1}{\sin 4x} = 0 \rightarrow 2\sin 2x - 1 = 0$

$\rightarrow \sin 2x = \dfrac{1}{2} \rightarrow 2x = 2k\pi + \dfrac{\pi}{6} \xrightarrow{\div 2} x = k\pi + \dfrac{\pi}{12}$ ①

$2x = 2k\pi + \dfrac{5\pi}{6} \xrightarrow{\div 2} x = k\pi + \dfrac{5\pi}{12}$ ② $\xrightarrow{x \in [0, \pi]} x = \left\{\dfrac{\pi}{12}, \dfrac{5\pi}{12}\right\}$

اختلاف جواب‌ها $\rightarrow \dfrac{5\pi}{12} - \dfrac{\pi}{12} = \dfrac{4\pi}{12} = \dfrac{\pi}{3} = \alpha \rightarrow \tan(2\alpha) = \tan\left(\dfrac{2\pi}{3}\right) = -\sqrt{3}$ ✓

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Year....... Month....... Day.......

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$\cos(x+\frac{\pi}{4}) = \cos x \cos\frac{\pi}{4} - \sin x \sin\frac{\pi}{4} = \frac{1}{\sqrt{6}} \Rightarrow \frac{1}{\sqrt{2}}(\cos x - \sin x) = \frac{1}{\sqrt{3}}$

$\to \cos x - \sin x = \frac{\sqrt{2}}{\sqrt{3}} = \frac{\sqrt{6}}{3} \quad (*)$

$m(\cos x - \sin x) - 3\sqrt{6}\sin 2x = \sqrt{6} \xrightarrow{*} m(\frac{\sqrt{6}}{3}) - 3\sqrt{6}\sin 2x = \sqrt{6}$

$\xrightarrow{\times\sqrt{6}} 2m - 18\sin 2x = 6$

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$\cos x - \sin x = \frac{\sqrt{6}}{3} \xrightarrow{\text{توان ۲}} \sin^2 x + \cos^2 x - \sin 2x = \frac{2}{3} \to 1 - \frac{2}{3} = \sin 2x \to \sin 2x = \frac{1}{3} \quad (**)$

$\xrightarrow{**} 2m - 18(\frac{1}{3}) = 6 \to 2m - 6 = 6 \to 2m = 12 \to m = 6$

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دلیل: $\cos(\alpha) = \cos(-\alpha) \to \cos(x-\frac{\pi}{3}) = \cos(+\frac{\pi}{3} - x)$

$\to \underbrace{\sin(x+\frac{\pi}{6})}_{\theta}\underbrace{\cos(\frac{\pi}{3}-x)}_{\beta} = 1 \xrightarrow[\text{طبق متمم بودن زوایا}]{\beta+\theta = 90^\circ} \sin\theta = \cos\beta$

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$\to \cos(x-\frac{\pi}{3})\cos(x-\frac{\pi}{3}) = 1 \to \cos^2(x-\frac{\pi}{3}) = 1 \to \cos(x-\frac{\pi}{3}) = \pm 1$

① $\cos(x-\frac{\pi}{3}) = 1 \to x - \frac{\pi}{3} = 2k\pi \to x = 2k\pi + \frac{\pi}{3}$ ✓

② $\cos(x-\frac{\pi}{3}) = -1 \to x - \frac{\pi}{3} = 2k\pi + \pi \to x = 2k\pi + \frac{4\pi}{3}$ ✓

$x \in [0, 2\pi] \quad x = \{\frac{\pi}{3}, \frac{4\pi}{3}\}$ → ۲ جواب دارد

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$\sin x + \sqrt{3}\cos x = \sqrt{2} \xrightarrow{\div 2} \frac{1}{2}\sin x + \frac{\sqrt{3}}{2}\cos x = \frac{\sqrt{2}}{2}$

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$\to \cos(\frac{\pi}{3})\sin x + \sin(\frac{\pi}{3})\cos x = \frac{\sqrt{2}}{2} \xrightarrow{\text{بسط } \sin(\alpha+\beta)} \sin(x+\frac{\pi}{3}) = \frac{\sqrt{2}}{2} = \sin(\frac{\pi}{4})$

① $x + \frac{\pi}{3} = 2k\pi + \frac{\pi}{4} \to x = 2k\pi - \frac{\pi}{12}$ ✓ $\xrightarrow{x\in[-\pi, 2\pi]} x = \{2\pi - \frac{\pi}{12}, \frac{5\pi}{12}, -\frac{\pi}{12}\}$

② $x + \frac{\pi}{3} = 2k\pi + \underbrace{\pi - \frac{\pi}{4}}_{\frac{3\pi}{4}} \to x = 2k\pi + \frac{5\pi}{12}$ ✓

$\frac{23\pi}{12} + \frac{5\pi}{12} - \frac{\pi}{12} = \frac{27\pi}{12} = \frac{9\pi}{4}$

مجموع جواب ها؟

→ ۳ جواب دارد

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دلیل: $\sin(\frac{3\pi}{2} - x) = -\cos x$

$\to 4\sin x(-\cos x) = 1 \to -4\cos x \sin x = 1 \to -2(2\sin x\cos x) = 1$

$\underbrace{\sin 2\alpha = 2\sin\alpha\cos\alpha}_{\text{دلیل}} \quad -2(\sin 2x) = 1 \to \sin 2x = -\frac{1}{2} = \sin(\frac{7\pi}{6})$

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① $2x = 2k\pi + \frac{7\pi}{6} \to x = k\pi + \frac{7\pi}{12}$ ✓ $\xrightarrow{x\in[0, 2\pi]} x = \{\frac{7\pi}{12}, \pi + \frac{7\pi}{12}, \pi - \frac{\pi}{12}, 2\pi - \frac{\pi}{12}\}$

② $2x = 2k\pi + \underbrace{\cancel{\pi - \frac{7\pi}{6}}}_{-\frac{\pi}{6}} \to x = k\pi - \frac{\pi}{12}$ ✓

جمع شان: $\frac{7\pi}{12} + \pi + \frac{7\pi}{12} + \pi - \frac{\pi}{12} + 2\pi - \frac{\pi}{12} = 4\pi + \pi = \boxed{5\pi}$ ✓

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می‌دانیم: $\cos^2\alpha = \frac{1+\cos 2\alpha}{2} \Rightarrow (1+\cos 2\alpha)(1+\cos 4\alpha)(1+\cos 8\alpha) = \frac{1}{8}$

$\to (2\cos^2\alpha)(2\cos^2 2\alpha)(2\cos^2 4\alpha) = \frac{1}{8} \to 8(\cos\alpha\cos 2\alpha\cos 4\alpha)^2 = \frac{1}{8}$

$\to (\cos\alpha\cos 2\alpha\cos 4\alpha)^2 = \frac{1}{64} \xrightarrow{\sqrt{\ }} \cos\alpha\cos 2\alpha\cos 4\alpha = \pm\frac{1}{8}$

$\xrightarrow{\frac{2\sin\alpha}{2\sin\alpha}} \frac{2\sin\alpha\cos\alpha\cos 2\alpha\cos 4\alpha}{2\sin\alpha} = \pm\frac{1}{8} \xrightarrow{\text{انقباض}} \frac{\sin 2\alpha\cos 2\alpha\cos 4\alpha}{2\sin\alpha} = \pm\frac{1}{8}$

$\xrightarrow{\times\frac{2}{2}} \frac{2\sin 2\alpha\cos 2\alpha\cos 4\alpha}{4\sin\alpha} = \pm\frac{1}{8} \xrightarrow{\text{انقباض}} \frac{\sin 4\alpha\cos 4\alpha}{4\sin\alpha} = \pm\frac{1}{8}$ 1.5

$\xrightarrow{\times\frac{2}{2}} \frac{2\sin 4\alpha\cos 4\alpha}{8\sin\alpha} = \pm\frac{1}{8} \xrightarrow{\text{انقباض}} \frac{\sin 8\alpha}{\sin\alpha} = \pm 1$ (I)

(1) $\to \frac{\sin 8\alpha}{\sin\alpha} = 1 \xrightarrow{\sin\alpha\neq 0} \sin 8\alpha = \sin\alpha \to 8\alpha = 2k\pi + \alpha \to \alpha = \frac{2k\pi}{7}$ ✓

$8\alpha = 2k\pi + \pi - \alpha \to \alpha = \frac{2k\pi}{9} + \frac{\pi}{9}$ ✓ (II)

(2) $\to \frac{\sin 8\alpha}{\sin\alpha} = -1 \to \sin 8\alpha = -\sin\alpha \xrightarrow{\sin(\pi+\alpha) = -\sin\alpha} \sin 8\alpha = \sin(\pi+\alpha)$

$8\alpha = 2k\pi + \pi + \alpha \to \alpha = \frac{2k\pi}{7} + \frac{\pi}{7}$ ✓ (III)

$8\alpha = 2k\pi + \cancel{\pi} - \cancel{\pi} - \alpha \to \alpha = \frac{2k\pi}{9}$ ✓ (IV)

$\xrightarrow[\text{(IV)}]{\text{(I), (II), (III)}} x \in [0, \pi] \overset{A}{\Rightarrow} x = \left\{ 0, \frac{2\pi}{7}, \frac{4\pi}{7}, \frac{6\pi}{7}, \frac{\pi}{9}, \frac{\pi}{3}, \frac{5\pi}{9}, \frac{7\pi}{9}, \pi, \frac{\pi}{7}, \frac{3\pi}{7}, \frac{5\pi}{7}, \frac{2\pi}{9}, \frac{4\pi}{9}, \frac{2\pi}{3}, \frac{8\pi}{9} \right\}$

max

★ A نمی‌تواند برابر خود π باشد چون طبق ★ $\sin\alpha$ را مخالف صفر در نظر گرفته‌ایم!

$\to$ ماکزیمم جواب معادله: π ✓

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$\tan 3x \cdot \tan x = 1 \to \tan 3x = \frac{1}{\tan x} = \cot x \xrightarrow{\cot x = \tan(\frac{\pi}{2} - x)}$

$\to \tan 3x = \tan(\frac{\pi}{2} - x) \to 3x = k\pi + \frac{\pi}{2} - x$ 1.5

$\to 4x = k\pi + \frac{\pi}{2} \to x = \frac{k\pi}{4} + \frac{\pi}{8}$

$\xrightarrow{x \in [\pi, 2\pi]} x = \left\{ \pi + \frac{\pi}{8}, \frac{5\pi}{4} + \frac{\pi}{8}, \frac{3\pi}{2} + \frac{\pi}{8} \right\} \xrightarrow{\text{جمع جواب‌ها}} \pi + \frac{\pi}{8} + \frac{5\pi}{4} + \frac{\pi}{8} + \frac{3\pi}{2} + \frac{\pi}{8}$

$= \frac{33\pi}{8}$ ✓

(7) عبارت را در $\frac{1}{2}$ ضرب می‌کنیم.

$$\frac{1}{2}\sin x + \frac{\sqrt{3}}{2}\cos x = \frac{\sqrt{2}}{2}$$

$$\cos 60^\circ \sin x + \sin 60^\circ \cos x = \frac{\sqrt{2}}{2} \longrightarrow \sin\left(x+\frac{\pi}{3}\right) = \frac{\sqrt{2}}{2}$$

$$\begin{cases} x+\frac{\pi}{3} = 2k\pi + \frac{\pi}{4} \longrightarrow x = 2k\pi - \frac{\pi}{12} \xrightarrow{-\pi \leq x \leq 2\pi} k=0,1 \rightarrow x = -\frac{\pi}{12}, \frac{23\pi}{12} \\ x+\frac{\pi}{3} = 2k\pi + \frac{3\pi}{4} \longrightarrow x = 2k\pi + \frac{5\pi}{12} \xrightarrow{-\pi \leq x \leq 2\pi} k=0 \rightarrow x = \frac{5\pi}{12} \end{cases}$$

$$S = \frac{23\pi}{12} - \frac{\pi}{12} + \frac{5\pi}{12} = \frac{27\pi}{12} = \boxed{\frac{9\pi}{4}}$$

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(10)

$$\tan 3x = \frac{1}{\tan x} = \cot x \longrightarrow \tan 3x = \tan\left(\frac{\pi}{2} - x\right)$$

$$3x = k\pi + \frac{\pi}{2} - x \longrightarrow 4x = k\pi + \frac{\pi}{2} \rightarrow x = \frac{k\pi}{4} + \frac{\pi}{8}$$

k	4	5	6	7
x	$\frac{9\pi}{8}$	$\frac{11\pi}{8}$	$\frac{13\pi}{8}$	$\frac{15\pi}{8}$

$$S = \left(\frac{9+11+13+15}{8}\right)\pi = \frac{48\pi}{8} = \boxed{6\pi}$$

(9)

$$1+\cos 2x = 2\cos^2 x$$

$$\begin{cases} 1+\cos 2\alpha = 2\cos^2\alpha \\ 1+\cos 4\alpha = 2\cos^2 2\alpha \\ 1+\cos 8\alpha = 2\cos^2 4\alpha \end{cases} ? \rightarrow 8\cos^2\alpha\,\cos^2 2\alpha\,\cos^2 4\alpha = \frac{1}{8}$$

$$\cos^2\alpha\,\cos^2 2\alpha\,\cos^2 4\alpha = \frac{1}{64} \xrightarrow{\sqrt{\ }} \cos\alpha\cos 2\alpha\cos 4\alpha = \frac{\pm 1}{8} \xrightarrow[\sin\alpha \neq 0]{\times \sin\alpha} \star$$

$$\underbrace{(\sin\alpha\cos\alpha)}_{\sin 2\alpha}\cos 2\alpha\cos 4\alpha = \pm\frac{\sin\alpha}{8} \rightarrow \frac{1}{2}\underbrace{(\sin 2\alpha\cos 2\alpha)}_{\frac{1}{2}\sin 4\alpha}\cos 4\alpha = \pm\frac{\sin\alpha}{8}$$

$$\frac{1}{2}\times\frac{1}{2}\underbrace{(\sin 4\alpha\cos 4\alpha)}_{\sin 8\alpha} = \pm\frac{\sin\alpha}{8} \rightarrow \frac{1}{\cancel{8}}\sin 8\alpha = \pm\frac{\sin\alpha}{\cancel{8}}$$

$$\sin 8\alpha = \pm\sin\alpha \rightarrow \begin{cases} 8\alpha = 2k\pi+\alpha \text{ یا } 2k\pi+\pi-\alpha \\ 8\alpha = 2k\pi-\alpha \text{ یا } 2k\pi+\pi+\alpha \end{cases}$$

$$\alpha = \frac{2k\pi}{7} \xrightarrow{\max A} k=3 \rightarrow x = \frac{6\pi}{7}$$

$$\alpha = \frac{2k\pi}{9} \xrightarrow{\max A} k=4 \rightarrow x = \frac{8\pi}{9}$$

$$\alpha = \frac{(2k+1)\pi}{7} \xrightarrow{\max A} k=2 \rightarrow x = \frac{5\pi}{7}$$

$$\alpha = \frac{(2k+1)\pi}{9} \xrightarrow{\max A} k=3 \rightarrow x = \frac{7\pi}{9}$$

$$\rightarrow \boxed{\max A = \frac{8\pi}{9}}$$

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